Transfer matrix solution of the Ising model on the Koch curve

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Abstract. A transfer matrix approach is used to calculate the partition function of the Ising model on the Koch curve. In zero magnetic field it is possible to obtain an exact analytic result. In the presence of a field the problem becomes more difficult. A procedure, based on an expansion about $H = 0$, is developed which shows that the spontaneous magnetisation vanishes identically and which makes it possible to give the zero field susceptibility as a power series in a small temperature dependent parameter.

It has recently been recognised that some phenomena of physical relevance, such as the threshold of percolating clusters and the behaviour of polymer chains, may be represented by self-similar structures called fractals (Mandelbrot 1977, 1982). In general, fractals are constructed in steps starting from a given shape, and rescaling it continuously down to a 'microscopic' length scale. The resulting geometrical shape, drawn in a d-dimensional Euclidean space, is characterised by a fractal (or Hausdorff) dimensionality D, which is usually different from d and non-integer.

Some preliminary investigations of critical phenomena on fractal lattices, via the renormalisation group technique, have been reported by Gefen *et al* (1980, 1983). In these works a decimation procedure is used to study the critical behaviour of an Ising model on a Koch curve with fractal dimensionality $D = \ln 4 / \ln 3$. In the present work we show that it is possible to use the transfer matrix technique to obtain exact analytic results for some thermodynamic functions of the Ising model on the Koch curve. If we restrict the spin interactions to nearest neighbours along the curve, the solution of the problem is trivial. However, if the spin interactions are between nearest neighbours in the plane, the decimation procedure is not exact, and the problem becomes an interesting example of the study of a system without translational symmetry.

The initial stages of the construction of the Koch curve are shown in figure 1. The spins on the lattice sites ($s_i = \pm 1, \forall i$) are assumed to interact with their nearest neighbours in the plane. There are two kinds of exchange parameters: (i) J, for nearest neighbours along the curve; (ii) and J', for nearest neighbours which do not correspond to bonds along the curve, as 2 and 4, 6 and 8, 8 and 10 etc, in figure 1. Then we may write the Ising Hamiltonian

$$\mathcal{H}_N = -J \sum_{i=1}^{4^N} s_i s_{i+1} - J' \sum_{l=1}^{N} \sum_{i=1}^{4^{l-1}} s_{4^{N-l}(4i-2)} s_{4^{N-l}(4i-2)+2} - H \sum_{i=1}^{4^N} s_i, \tag{1}$$
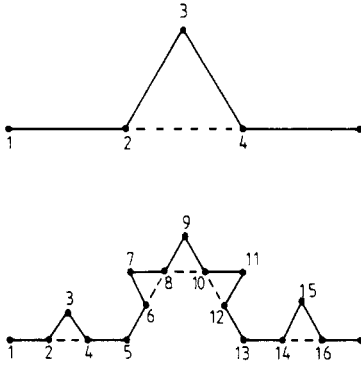
Figure 1. The first two stages of the construction of the Koch curve. Spins are placed on each site of the curve and interactions are allowed between nearest neighbours only. Broken lines indicate those interactions with exchange parameter J'.

where N indicates the Nth step of the construction, $\hat{N} = 4^N$ is the total number of spins, H is the applied magnetic field, and we assume periodic boundary conditions in each stage of the construction. It should be remarked that the Koch curve, although embedded in a two-dimensional Euclidean space, is still one-dimensional from the topological point of view. Indeed, it is possible to isolate finite portions of the curve by the removal of two sites only. Also, there is no translational symmetry, and the fractal character of the lattice clearly governs the places where an interaction J' should be added.

If we define the matrix elements

$$\langle s|P|s'\rangle = \exp(\beta J' ss'), \tag{2}$$

and

$$\langle s|R|s'\rangle = \exp[\beta Jss' + \tfrac{1}{2}\beta H(s+s')], \tag{3}$$

where $\beta = (kT)^{-1}$, the partition function of the first stage of the construction is given by

$$Z(T, H, \hat{N} = 4) = \sum_{\{s\}} \exp(-\beta \mathcal{H}_1)$$

$$= \sum_{s_2, s_4} \langle s_2|R^2|s_4\rangle\langle s_2|P|s_4\rangle\langle s_4|R^2|s_2\rangle. \tag{4}$$

Using the definitions

$$\langle s|Q^2|s'\rangle = \langle s|R^2|s'\rangle\langle s|P|s'\rangle, \tag{5}$$

and $R_1 = Q^2 R^2$, we have $Z(T, H, 4) = \text{Tr } R_1$. In the next step of the construction we have $Z(T, H, \hat{N} = 16) = \text{Tr } R_2$, where $R_2 = Q^2 R^2 Q^2 Q^2 Q^2 R^2 Q^2 R^2$. In general, it is easy to see that for all N we have $Z(T, H, 4^N) = \text{Tr } R_N$, where R_N is given by the product of the matrices R^2 and Q^2 in a well defined sequence, according to the presence of pair interactions along bonds which may or may not belong to the curve. Of course, this form of R_N reflects the lack of translational invariance of the Koch curve.

In zero field the matrices Q^2 and R^2 commute and can be diagonalised simultaneously. The trace of R_N may be written as

$$\text{Tr } R_N = \eta_1^{a-}\lambda_1^{a-} + \eta_2^{a-}\lambda_2^{a-} \tag{6}$$

where η_1 and η_2, and λ_1 and λ_2, are the eigenvalues of Q^2 and R^2 respectively, and the subscript 1 denotes the largest eigenvalues of these matrices. The exponents $a_\pm$, which are given by

$$a_\pm = 4^{N-1}\left(1 \pm \sum_{i=1}^{N-1} (\tfrac{1}{4})^i\right),$$
(7)

count the number of times the matrices Q^2 and R^2 appear in the product R_N. In the thermodynamic limit it is easy to calculate the free energy per spin

$$f(T, H = 0) = -\tfrac{1}{6}kT \ln(\eta_1^2 \lambda_1),$$
(8)

where

$$\eta_1 = 2 \, e^{\beta J'}\left(\cosh 2\beta J + e^{-2\beta J'}\right),$$
(9)

and

$$\lambda_1 = 4 \cosh^2 \beta J.$$
(10)

For $J, J' > 0$, the ground state is ferromagnetic. In the presence of competing interactions, however, the ground state may become antiferromagnetic and highly degenerate. For $J > 0$, and $-J' > J$, there is a residual entropy per spin given by $s = \tfrac{1}{3}k \ln 2$.

In the presence of an applied field the matrices Q^2 and R^2 do not commute. Moreover, since the fractals are not translationally invariant, this makes it very difficult to find an expression for the free energy $f(T, H)$. However, it is possible to establish the recursion relation

$$R_N = R_{N-1}^2 S R_{N-1}^2$$
(11)

where $S = R^{-2}Q^2$. In general, it is awkward to write the eigenvalues of R_{N-1} in terms of those of R_N and S, although this is not difficult if we perform expansions for small H and keep terms up to order H^2. In the former case we are faced with quite untractable nonlinear difference equations, while in the latter case the equations are linear. In the remainder of this paper we show how this can be done and obtain, as an example, the zero field susceptibility as a power series in a small temperature dependent parameter. As it should be expected, the spontaneous magnetisation vanishes and the susceptibility is well behaved except at $T = 0$.

If we expand about $H = 0$, the matrices Q^2 and R^2 may be written as

$$A = C + \beta H D + \beta^2 H^2 E + O(H^3),$$
(12)

where C and E are matrices of the form

$$C = \begin{pmatrix} c_1 & c_2 \\ c_2 & c_1 \end{pmatrix},$$
(13)

with eigenvalues $\sigma_{1,2} = c_1 \pm c_2$, while D has the form

$$D = \begin{pmatrix} d_1 & d_2 \\ -d_2 & -d_1 \end{pmatrix},$$
(14)

with eigenvalues $\delta_{1,2} = \pm(d_1^2 - d_2^2)^{1/2}$. Thus, the eigenvalues of A, up to order H^2, are given by

$$\alpha_{1,2} = \sigma_{1,2} + \beta^2 H^2 [\varepsilon_{1,2} \pm \delta^2/(\sigma_1 - \sigma_2)],$$
(15)

where $\varepsilon_{1,2}$ are the eigenvalues of E, and $\delta^2 = \delta_1^2 = \delta_2^2$. If we notice that the product of matrices of the form given by equation (12) still keeps the same form up to terms of order H^2, it is possible to write R_N as in equation (12). So, the eigenvalues of R_N are given by equation (15) with all quantities denoted by an additional superscript N. The matrix S, on the other hand, may be explicitly written in the form

$$S = S_0 + \beta H S_1 + \beta^2 H^2 S_2 + O(H^3), \tag{16}$$

with eigenvalues $\tau_{1,2}$ given by

$$\tau_{1,2} = \phi_{1,2} + \beta^2 H^2 [\omega_{1,2} \pm \gamma^2/(\phi_1 - \phi_2)], \tag{17}$$

where

$$\phi_{1,2} = [e^{\beta J'}(e^{2\beta J} + e^{-2\beta J}) \pm 2 e^{-\beta J'}]/(e^{\beta J} \pm e^{-\beta J})^2,$$

$$\gamma_{1,2} = \pm\sqrt{\gamma^2} = \pm 2i\, e^{2\beta J}(e^{\beta J'} - e^{-\beta J'})/(e^{2\beta J} - e^{-2\beta J})^2, \tag{18}$$

$$\omega_{1,2} = (e^{\beta J'} - e^{-\beta J'})(4 \mp 5\, e^{2\beta J} \mp e^{-2\beta J})/(e^{2\beta J} - e^{-2\beta J})^2, \tag{19}$$

are the eigenvalues of the matrices S_0, S_1, and S_2, respectively. At this point we are prepared to undertake the calculations to obtain recursion relations for the eigenvalues of R_{N+1} in terms of those of R_N and S. It is then straightforward to show that

$$\sigma_{1,2}^{(N+1)} = \phi_{1,2}[\sigma_{1,2}^{(N)}]^4, \tag{20}$$

$$[\delta^{(N+1)}]^2 = \gamma^2[\sigma_1^{(N)}\sigma_2^{(N)}]^4 + 2i\gamma_1 d_2^{(N)}[\sigma_1^{(N)}\sigma_2^{(N)}]^2[\sigma_1^{(N)} + \sigma_2^{(N)}]$$
$$\times \{\phi_1[\sigma_1^{(N)}]^2 + \phi_2[\sigma_2^{(N)}]^2\} + [\sigma_1^{(N)} + \sigma_2^{(N)}]^2\{\phi_1[\sigma_1^{(N)}]^2$$
$$+ \phi_2[\sigma_2^{(N)}]^2\}^2[\delta^{(N)}]^2, \tag{21}$$

and

$$\varepsilon_{1,2}^{(N+1)} = 4\phi_{1,2}\varepsilon_{1,2}^{(N)}[\sigma_{1,2}^{(N)}]^3 + 2\phi_{1,2}[\sigma_{1,2}^{(N)}]^2[\delta^{(N)}]^2$$
$$+ \omega_{1,2}[\sigma_{1,2}^{(N)}]^4 + (\sigma_1^{(N)} + \sigma_2^{(N)})\{(\sigma_1^{(N)} + \sigma_2^{(N)})\phi_{2,1}[\delta^{(N)}]^2$$
$$+ 2i\gamma_1 d_2^{(N)}[\sigma_{1,2}^{(N)}]^2\}. \tag{22}$$

However, we should still eliminate from these equations the matrix element $d_2^{(N)}$ of D_N. This is finally accomplished by the relation

$$d_2^{(N+1)} = 2i\gamma_1[\sigma_1^{(N)}\sigma_2^{(N)}]^2$$
$$+ [\sigma_1^{(N)} + \sigma_2^{(N)}]\{\phi_1[\sigma_1^{(N)}]^2 + \phi_2[\sigma_2^{(N)}]^2\}d_2^{(N)}. \tag{23}$$

From equations (20)–(23) we formally obtain the eigenvalues of R_N, for all values of N, as functions of the known quantities $\phi_{1,2}$, $\omega_{1,2}$, γ^2, $\sigma_{1,2}^{(0)}$, $\varepsilon_{1,2}^{(0)}$ and $[\delta^{(0)}]^2$. Indeed, it is quite simple to write the partition function in the form

$$Z(T, H, 4^N) = \sigma_1^{(N)}[1 + cz^{4^N} + 4^N \beta^2 H^2(v_1^{(N)} + v_2^{(N)})] + O(H^3), \tag{24}$$

where

$$\sigma_1^{(N)} = \phi_1^{-1/3}[\phi_1^{1/3}\sigma_1^{(0)}]^{4^N}, \tag{25}$$

$$c = (\phi_1/\phi_2)^{1/3}, \tag{26}$$

$$z = \frac{\phi_2^{1/3}\sigma_2^{(0)}}{\phi_1^{1/3}\sigma_1^{(0)}} = \left(\tanh \beta J \frac{\cosh 2\beta J - e^{-2\beta J'}}{\cosh 2\beta J + e^{-2\beta J'}}\right) \tag{27}$$

and

$$v_1^{(N)} = \frac{\varepsilon_1^{(0)}}{\sigma_1^{(0)}} + \sum_{l=0}^{N-1} \frac{1}{4^{l+1}} [u_l[\sigma_1^{(0)}]^{-2}(2 + c^{-3} + 2c^{-2}z^{4^l} + c^{-1}z^{2\times4^l})$$

$$+ \omega_1 \phi_1^{-1} + 2\gamma^2 \phi_1^{-2} t_l c^2 z^2 (1 + cz^{4^l})] \tag{28}$$

$$v_2^{(N)} = \frac{cz^{4^N} \varepsilon_2^{(0)}}{\sigma_2^{(0)}} + \sum_{l=0}^{N-1} \frac{cz^{4^N - 4^{l+1}}}{4^{l+1}}$$

$$\times [u_l[\sigma_1^{(0)}]^{-2} c^{-1}(1 + 2cz^{4^l} + 2c^{-1}z^{2\times4^l} + c^2 z^{2\times4^l}) + \omega_2 \phi_2^{-1} z^{4^{l+1}}$$

$$+ 2\gamma^2 c^3 z^2 \phi_1^{-2} t_l z^{2\times4^l} (1 + cz^{4^l})]. \tag{29}$$

In equations (27) and (28) we have

$$t_l = \sum_{k=0}^{l-1} z^{2(4^k - l)} \prod_{p=k+1}^{l-1} (1 + cz^{4^p} + c^{-1}z^{2\times4^p} + z^{2\times4^p}), \tag{30}$$

$$u_l = [\delta^{(0)}]^2 q_{l-1} + \gamma^2 [\sigma_1^{(0)}]^2 \phi_1^{-2} c^4 \sum_{k=0}^{l-1} \frac{q_{l-1}}{q_k} (2z^2 t_{k+1} z^{2\times4^k} - z^{4^{k+1}}), \tag{31}$$

$$q_l = \prod_{k=0}^{l} (1 + 2cz^{4^k} + (c^2 + 2c^{-1})z^{2\times4^k} + 4z^{3\times4^k} + (c^{-2} + 2c)z^{4\times4^k} + 2c^{-1}z^{5\times4^k} + z^{6\times4^k}). \tag{32}$$

Since the partition function is given as a power series in the variable $z \leqslant 1$, we do not expect any non-analytic behaviour, except at $T = 0$, when $z = 1$. Indeed, the free energy per spin in the thermodynamic limit is given by

$$f(T, H) = -kT\left\{\ln(\phi_1^{1/3}\sigma_1^{(0)}) + \lim_{N\to\infty} \frac{1}{4^N} \ln[1 + 4^N\beta^2 H^2(v_1^{(N)} + v_2^{(N)})]\right\} + O(H^3). \tag{33}$$

The magnetisation is

$$m(T, H) = \lim_{N\to\infty} \frac{2\beta H(v_1^{(N)} + v_2^{(N)})}{1 + 4^N\beta^2 H^2(v_1^{(N)} + v_2^{(N)})} + O(H^3), \tag{34}$$

which vanishes identically for $H = 0$. On the other hand, the zero field susceptibility is given by

$$\chi(T, H = 0) = \lim_{N\to\infty} 2\beta(v_1^{(N)} + v_2^{(N)}) = 2\beta v_1 \tag{35}$$

where $v_1 = \lim_{N\to\infty} v_1^{(N)}$ is a well behaved function of temperature, and $\lim_{N\to\infty} v_2^{(N)} = 0$.

In conclusion, we have used the transfer matrix technique to obtain an exact analytic expression for the free energy per spin of the zero field Ising model on a Koch curve. For $H \neq 0$, the transfer matrix for the whole curve is written as a non-periodic n-fold product of non-commuting matrices, and the problem of finding its largest eigenvalues becomes quite difficult. We show, however, that the spontaneous magnetisation vanishes and the zero field susceptibility may be written as a power series in a small temperature dependent parameter.

References

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